$\star$ $E = \{1, x, x^2\} \subseteq P_2(\mathbb{R})$

$$L(a + bx + cx^2) = (a+b+c) + (a+b)x + (a+b)x^2$$

Compute $\mathrm{Rep}_{E,E}(L)$.

$\hookrightarrow \mathrm{Rep}_{B,D}(L)$

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ \cup\!\!| & & \cup\!\!| \\ B & & D \end{array}$$

$$\mathrm{Rep}_{B,D}(L) = \left[[L(b_1)]_D \,\middle|\, [L(b_2)]_D \,\middle|\, \cdots \,\middle|\, [L(b_n)]_D \right]$$

Want $\mathrm{Rep}_{E,E}(L) = \left[[L(1)]_E \,\middle|\, [L(x)]_E \,\middle|\, [L(x^2)]_E \right]$

NB: $L(1) = L(1\cdot 1 + 0\cdot x + 0\cdot x^2) = (1+0+0) + (1+0)x + (1+0)x^2$
$= 1\cdot 1 + 1\cdot x + 1\cdot x^2$

$\star$ $L(a + bx + cx^2) = (a+b+c) + (a+b)x + (a+b)x^2$

$L(x) = L(0\cdot 1 + 1\cdot x + 0\cdot x^2) = 1 + x + x^2$

$L(x^2) = L(0\cdot 1 + 0\cdot x + 1\cdot x^2) = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$$[L(1)]_E = \begin{array}{ll} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{array}{l} \text{Coeff of } b_1 = 1 \\ \text{Coeff of } b_2 = x \\ \text{Coeff of } b_3 = x^2 \end{array} \end{array}$$

$\star$

$$[L(x)]_E = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad [L(x^2)]_E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{Rep}_{E,E}(L) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Remark: this was easy because E was a very nice basis...

In general, we need to use:

$$\begin{tikzcd}
V_B \arrow[r, "\mathrm{Rep}_{B,D}(L)"] \arrow[d, "\mathrm{Rep}_{B,B'}(id)"'] & W_D \arrow[d, "\mathrm{Rep}_{D,D'}(id)"] \\
V_{B'} \arrow[r, "\mathrm{Rep}_{B',D'}(L)"'] & W_{D'}
\end{tikzcd}$$

These matrices Describe coord changes.

$$\mathrm{Rep}_{B,D}(L) = \left[\mathrm{Rep}_{D',D}(id)\right]\mathrm{Rep}_{B',D'}(L)\cdot\left[\mathrm{Rep}_{B,B'}(id)\right]$$

$$\begin{tikzcd}
P_2(\mathbb{R})_B \arrow[r, "\boxed{\mathrm{Rep}_{B,D}(L)}"] \arrow[d, "\mathrm{Rep}_{B,E}(id)"'] & P_2(\mathbb{R})_D \arrow[d, "\mathrm{Rep}_{D,E}(id)"] \\
P_2(\mathbb{R})_E \arrow[r, "\mathrm{Rep}_{E,E}(L)"'] & P_2(\mathbb{R})_E
\end{tikzcd}$$

$$\mathrm{Rep}_{E,E}(L) = \begin{bmatrix} | & | & 0 \\ | & | & 0 \end{bmatrix} \leftarrow \text{was easy to compute}$$

Orthogonal Complement:

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 : \begin{array}{l} 2a-b+c=0 \\ a-b-d=0 \end{array} \right\}$$

$$W = \mathrm{Col}(A) \implies W^\perp = \mathrm{null}(A^T)$$

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{l} \boxed{b = 2a+c} \\ d = a-b \end{array} \right\}$$

$$d = a - b = a - (2a+c)$$

$$= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{l} b = 2a+c \\ d = -a-c \end{array} \right\} = \left\{ \begin{bmatrix} a \\ 2a+c \\ c \\ -a-c \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} a \\ 2a \\ 0 \\ -a \end{bmatrix} + \begin{bmatrix} 0 \\ c \\ c \\ -c \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

$$= \left\{ a \underline{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}} + c \underline{\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}} : a, c \in \mathbb{R} \right\}$$

$$= \operatorname{Col} \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}} = \operatorname{Col}(A)$$

$$\therefore \; W^{\perp} = \operatorname{null}(A^{T})$$

$$= \operatorname{null} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$= \operatorname{null} \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \\ \begin{matrix} a & \;\, b & \;\, c & \;\, d \end{matrix} \end{array}$$

$$= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a - 2c + d = 0 \\ b + c - d = 0 \end{matrix} \right\}$$

→ Keep going... give a basis...

$$\boxed{W = \operatorname{Col}(A) \implies W^{\perp} = \operatorname{null}(A^{T}).}$$

Ex: Show $B = \{1+x,\ 1+2x+x^2,\ \underset{\substack{\shortparallel \\ 3x}}{x+2x}\}$ spans $P_2(\mathbb{R})$.

Sol: $$[B]_E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix} = M$$

$$\operatorname{RREF}(M) = \operatorname{RREF}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} = \operatorname{RREF}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} = \operatorname{RREF}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$:)

So because $\operatorname{rank}(M) = \operatorname{rank}(\operatorname{RREF}(M)) = 3$, and $\dim(P_2(\mathbb{R})) = 3$, B spans. ☑

Ex: Apply Gram-Schmidt process to $v_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix}$.

Sol: $u_1 = v_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$

$$u_2 = v_2 - \boxed{\mathrm{proj}_{u_1}(v_2)} = v_2 - \boxed{\frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1}$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix}}{\begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} - \frac{-15 - 9 - 18 - 3}{9 + 1 + 4 + 1} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} + \frac{3(5 + 3 + 6 + 1)}{15} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$$

$\therefore$ $u_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$ span same space as v_1, v_2

and they are orthogonal :) $\square$

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}$$

$\lambda = 2$ $\quad$ $\lambda = 1$ $\quad$ $\lambda = -5$ $\quad$ $\lambda = \pi$

To make orthonormal:

Add step: Normalize each vector $\rightsquigarrow$ $u_1 \rightsquigarrow w_1 = \frac{1}{|u_1|} u_1$, $u_2 \rightsquigarrow w_2 = \frac{1}{|u_2|} u_2$

In the previous example: $u_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$

$$|u_1| = \sqrt{3^2+1^2+2^2+1^2} = \sqrt{15}, \quad |u_2| = \sqrt{4^2+6^2+3^2} = \sqrt{5^2+6^2} = \sqrt{61}$$

$$\therefore \; w_1 = \frac{1}{|u_1|} u_1 = \frac{1}{\sqrt{15}} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \quad w_2 = \frac{1}{|u_2|} u_2 = \frac{1}{\sqrt{61}} \begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$$

is orthonormal collection spanning same space as v_1, v_2. $\boxed{\checkmark}$

$$\begin{tikzcd} V_B \arrow[r, "\mathrm{Rep}_{B,D}(L)"] \arrow[d, "\sim"'] & W_D \arrow[d, "\sim"] \\ \mathbb{R}^{\dim(V)} \arrow[r, "M = \mathrm{Rep}_{B,D}(L)"'] & \mathbb{R}^{\dim(W)} \end{tikzcd}$$

$L : V \to W$ isomorphism iff L^{-1} exists.

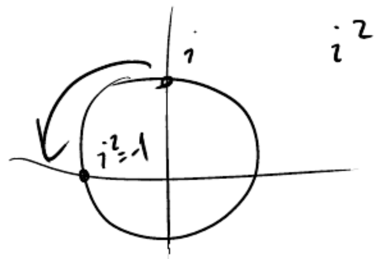
* $L(B) = X$ ← basis of W iff L is an iso.

* $L^{-1} : W \to V$ via $\underline{L^{-1}(L(b)) = b}$.

extend linearly (using lin. ext prop. from before Exam 2).

Ex: $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has e-values $\lambda = \pm i$ (check).

(is $\underline{\text{NOT}}$ diagonalizable over $\mathbb{R}$, but $\underline{\text{IS}}$ diagonalizable over $\mathbb{C}$



$i^2 = -1$

$\mathbb{C}$

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a+b=0 \right\}$$